CONSTRUCTION OF A SYSTEM OF HOMOGENEOUS SOLUTIONS AND ANALYSIS OF THE ROOTS OF THE DISPERSION EQUATION OF ANTISYMMETRIC VIBRATIONS OF A PIEZOELECTRIC PLATE

V. V. Madorskii and Yu. A. Ustinov

UDC 539.3:534.1

INTRODUCTION

In a number of papers (for example, [1, 2]) vibration of plates (slabs) of piezoelectric materials is investigated on the basis of approximate two-dimensional equations. The area of application of these is virtually not studied at all. However, from the character of their structure it emerges that they are poorly adapted for the investigation of high-frequency vibration of thin and thick plates; it is not capable of describing an edge resonance; and so forth. Comparisons with experiments [3] sometimes give considerable discrepancies even for fairly low frequencies. The investigation of the area of application of such two-dimensional theories (for example, with respect to the parameter of relative thickness $\varepsilon = h/a$ and the frequency Ω) requires analysis of the problem on the basis of three-dimensional equations of electroelasticity. In the classical theory of elasticity an analogous problem was analyzed on the basis of combined application of the theory of homogeneous solutions and the asymptotic method [4].

As we know, in dynamic problems the construction of homogeneous solutions for plates (slabs) is connected with the determination of the roots of the dispersion equation. The dispersion equation in the case of considering a plane problem of electroelastic vibration under the condition that the vector of external forces is zero on the faces of the plate, and that the normal constituents of electrical induction and the tangential constituents of the electrical field on the boundary between the piezoelectric material and vacuum are equal, was obtained in [5]. By means of an approximate solution of this equation, the dependence of the phase velocity on the frequency was found for the first two lowest modes of symmetric and antisymmetric types of vibration, respectively. For a circular piezoactive waveguide the real roots of the dispersion equation were investigated in [6] for normal waves of axisymmetric type.

The present paper is devoted to the construction of a system of homogeneous solutions of antisymmetric vibrations of a piezoelectric plate with an arbitrary plan view.

§1. We denote by A the region occupied by a cylinder (plate); by Γ, its side surface; by S, the middle surface; by 2h, the thickness; and by b, a characteristic linear dimension of the plate. We refer A to a Cartesian coordinate system (x_1, x_2, x_3) with the origin at S and the x_3 axis parallel to the generator of the cylinder. We assume that the material of the plate is a preliminarily polarized ceramic of the class $C_{6V} \equiv 6mm$ with the polarization axis parallel to the x_3 axis. The ends of the plate are completely covered by electrodes of infinitely small thickness which are shortened.

Making use of the relations [2], we can write the vibration equations of the plate in the form

$$\begin{gathered}
(\rho\omega^2 + c_{11}\Delta + c_{66}\partial_2^2 - c_{11}\partial_2^2 + c_{44}\partial_3^2)\, u_1 + (c_{12} + c_{66})\,\partial_1\partial_2 u_2 + (c_{44} + c_{13})\,\partial_1\partial_3 u_3 + (e_{31} + e_{15})\,\partial_1\partial_3\varphi = 0;\\
(\rho\omega^2 + c_{66}\Delta + c_{44}\partial_3^2)\, u_2 + (c_{66} + c_{12})\,\partial_1\partial_2 u_1 + (c_{11} - c_{66})\,\partial_2^2 u_2 + (c_{44} + c_{13})\,\partial_2\partial_3 u_3 + (e_{31} + e_{15})\,\partial_2\partial_3\varphi = 0;\\
(c_{44} + c_{13})(\partial_1\partial_3 u_1 + \partial_2\partial_3 u_2) + (\rho\omega^2 + c_{44}\Delta + c_{33}\partial_3^2)\, u_3 + (e_{15}\Delta + e_{33}\partial_3^2)\,\varphi = 0; \qquad (1.1)\\
(e_{15} + e_{31})(\partial_1\partial_3 u_1 + \partial_2\partial_3 u_2) + (e_{15}\Delta + e_{33}\partial_3^2)\, u_3 - (\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2)\,\varphi = 0;\\
\partial_i = \frac{\partial}{\partial x_i};\ \Delta = \partial_1^2 + \partial_2^2,
\end{gathered}$$

Rostov-on-Don. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 138-145, November-December, 1976. Original article submitted December 30, 1975.

where $U_i = u_i e^{j\omega t}$ are the components of the displacement vector; φ is the potential of the electrical field connected with the voltage vector by the relation $\mathbf{E} = -\mathrm{grad}\,\varphi$; c_{mn} are the elastic constants; e_{mn} are the piezoconstants; and ε_{mn} are the dielectrical permittivities.

The following conditions are given on the ends at $x_3 = \pm h$:

$$t_{3i}(x, \pm h) = 0;\ \varphi(x, \pm h) = C;\ x = \{x_1, x_2\}, \tag{1.2}$$

where t_{ni} are the amplitudes of the mechanical stresses; $C \equiv$ const.

We shall assume that $C = 0$, since it is not difficult to set up a particular solution of Eqs. (1.1) which does not depend on x and satisfies the condition $C \neq 0$. The solutions of Eqs. (1.1) which satisfy the homogeneous boundary conditions (1.2) are called homogeneous solutions.

Invariance of Eqs. (1.1) and the boundary conditions (1.2) relative to the rotation group about x_3 allows us to simplify the original problem. Indeed, we represent the plane field $\{u_1, u_2\}$ in the form

$$u_1 = \partial_1\Phi + \partial_2\Psi;\ u_2 = \partial_2\Phi - \partial_1\Psi. \tag{1.3}$$

It is obvious that such representation is always possible. Substitution of (1.3) into (1.1) allows us to separate the original problem into the following two:

$$L(\Delta)\mathbf{V} = 0,\quad \{M(\Delta)\mathbf{V}\}_{x_3=\pm h} = 0; \tag{1.4}$$

$$H(\Delta)\Psi \equiv (c_{44}\partial_3^2 + c_{66}\Delta + \rho\omega^2)\Psi = 0;\quad \{\partial_3\Psi\}_{x_3=\pm h} = 0, \tag{1.5}$$

where $\mathbf{V} = \{\Phi, u_3, \varphi\}$ is a vector function; L, M are matrix operators of the form

$$L = \begin{pmatrix} \rho\omega^2 + c_{11}\Delta + c_{44}\partial_3^2 & (c_{44} + c_{13})\partial_3 & (e_{15} + e_{31})\partial_3 \\ \partial_3(c_{44} + c_{13})\Delta & \rho\omega^2 + c_{33}\partial_3^2 + c_{44}\Delta & e_{15}\Delta + e_{33}\partial_3^2 \\ \partial_3(e_{15} + e_{31})\Delta & e_{15}\Delta + e_{33}\partial_3^2 & -(\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2) \end{pmatrix},$$

$$M = \begin{pmatrix} c_{44}\partial_3 & c_{44} & e_{15} \\ c_{11}\Delta & c_{33}\partial_3 & e_{33}\partial_3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$

We introduce the concept of elementary solutions of the first and second kinds. For this we shall seek the solutions of the problems (1.4), (1.5) in the form

$$\begin{gathered} V = v(x_3)m(x);\ (\Delta + a^2)m = 0;\ v = \{v_1, v_2, v_3\}; \\ \Psi = \psi(x_3)n(x);\ (\Delta + k^2)n = 0. \end{gathered} \tag{1.7}$$

We introduce the dimensionless coordinates and quantities according to the expression

$$\begin{gathered} \zeta = x_3/h;\ \eta = x_2/b;\ \xi = x_1/b;\ \varepsilon = h/b; \\ \Omega^2 = \rho\omega^2 h^2/c_{44};\ \alpha = ah;\ \lambda = kh. \end{gathered} \tag{1.8}$$

Substituting (1.7) into (1.4), (1.5) and taking into account (1.8), after separation of the variables we obtain the spectral problem

$$L(-\alpha^2)v = 0;\ \{M(-\alpha^2)v\}_{\zeta=\pm 1} = 0; \tag{1.9}$$

$$H(-\lambda^2)\psi = 0;\ \{\psi_{,\zeta}\}_{\zeta=\pm 1} = 0. \tag{1.10}$$

As will be seen below, the problems (1.9), (1.10) have a discrete spectrum. Let $\{\alpha_k\}$ be the spectrum of the problem (1.9), while $\{\lambda_j\}$ is the spectrum of the problem (1.10). Each of the points of the spectrum is matched by solutions of the form

$$V_k = v_k m_k;\ (\varepsilon^2\nabla^2 + \alpha_k)m_k = 0;\ \nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2; \tag{1.11}$$

$$\Psi_j = \psi_j n_j;\ (\varepsilon^2\nabla^2 + \lambda_j)n_j = 0. \tag{1.12}$$

The solution (1.11) is called an elementary solution of the first kind, while (1.12) is an elementary solution of the second kind. In total they determine the complete system of homogeneous solutions which allows us to satisfy arbitrary boundary conditions on the side surface.

§2. We consider the spectral problems (1.9), (1.10). The problem (1.10) is solved simply and is described by the system of relations

$$\begin{gathered}\psi_j = h^{-1/2}\cos \gamma_j x_3;\ \sin \gamma_j h = 0;\ j = 0, 2, 4, \ldots;\\ \psi_j = h^{-1/2}\sin \gamma_j x_3;\ \cos \gamma_j h = 0,\ j = 1, 3, 5, \ldots;\\ \gamma_j^2 = c_{44}^{-1}\left(c_{66}k_j^2 - \rho\omega^2\right).\end{gathered} \tag{2.1}$$

The system of eigenfunctions (2.1) is a complete orthonormalized system.

Much more difficult is the problem (1.9), whose symbolic form contains a system of three ordinary differential equations of the second order with constant coefficients and homogeneous boundary conditions. The general solution of the system of equations has the form

$$\begin{gathered}v_1 = \sum_{i=1}^{3} h_i\beta_i\left[A_i \operatorname{sh}(\beta_i\zeta) + B_i \operatorname{ch}(\beta_i\zeta)\right];\\ v_2 = \sum_{i=1}^{3} e_i\left[A_i \operatorname{ch}(\beta_i\zeta) + B_i \operatorname{sh}(\beta_i\zeta)\right];\\ v_3 = \sum_{i=1}^{3}\left[A_i \operatorname{ch}(\beta_i\zeta) + B_i \operatorname{sh}(\beta_i\zeta)\right].\end{gathered} \tag{2.2}$$

where A_i and B_i are arbitrary constants; β_i are the roots of the characteristic equation

$$\begin{gathered}\beta^6 + P\beta^4 + Q\beta^2 + R = 0;\\ P = N_1\Omega^2 - N_2\alpha^2;\quad Q = D_1\Omega^4 - D_2\Omega^2\alpha^2 + D_3\alpha^4;\\ R = -\alpha^2\left(C_1\Omega^4 - C_2\Omega^2\alpha^2 + C_3\alpha^4\right);\\ N_1 = c_{44}\left[(c_{33} + c_{44})\varepsilon_{33} + e_{33}^2\right]/G;\quad D_1 = c_{44}^2\varepsilon_{33}/G;\\ N_2 = \left[c_{33}(c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11}) + c_{33}(e_{31} + e_{15})^2 + c_{11}e_{33}^2 - 2c_{44}(e_{31}e_{33} + c_{13}\varepsilon_{33}) -\right.\\ \left. - 2c_{13}e_{33}(e_{15} + e_{31}) - c_{13}^2\varepsilon_{33}\right]/G;\quad G = c_{44}\left(\varepsilon_{33}c_{33} + e_{33}^2\right);\\ D_2 = c_{44}\left[(c_{44} + c_{11})\varepsilon_{33} + (c_{44} + c_{33})\varepsilon_{11} + (e_{31} + e_{15})^2 + 2e_{15}e_{33}\right]/G;\\ D_3 = \left[c_{44}\left(c_{11}\varepsilon_{33} + e_{31}^2\right) + c_{11}(c_{33}\varepsilon_{11} + 2e_{15}e_{33}) - c_{13}^2\varepsilon_{11} - 2c_{13}\times\right.\\ \left.\times\left(c_{44}\varepsilon_{11} + e_{31}e_{15} + e_{15}^2\right)\right]/G;\quad C_1 = c_{44}^2\varepsilon_{11}/G;\\ C_2 = \left[(c_{44} + c_{11})\varepsilon_{11} + e_{15}^2\right]c_{44}/G;\\ C_3 = c_{11}\left(c_{44}\varepsilon_{11} + e_{15}^2\right)/G;\\ e_i = \frac{(c_{44} + c_{13})\left(\varepsilon_{33}\beta_i^2 - \varepsilon_{11}\alpha^2\right) + (e_{31} + e_{15})\left(e_{33}\beta_i^2 - e_{15}\alpha^2\right)}{(c_{44} + c_{13})\left(e_{33}\beta_i^2 - e_{15}\alpha^2\right) - (e_{31} + e_{15})\left(c_{33}\beta_i^2 + c_{44}\Omega^2 - c_{44}\alpha^2\right)};\\ h_i = -\frac{(c_{44} + c_{13})e_i + (e_{31} + e_{15})}{c_{44}\left(\Omega^2 + \beta_i^2\right) - c_{11}\alpha^2}.\end{gathered} \tag{2.3}$$

It should be noted that the problem (1.10) is purely elastic and its solution (2.1) does not contain the characteristics of the electrical field, while the problem (1.9) is coupled and in the expressions (2.2), (2.3) there are all constants of the material. The coupling of the problem in fact determines its complexity.

The general solution (2.2) decomposes into a symmetric solution $A_i = 0$ and an antisymmetric solution $B_i = 0$. We consider the antisymmetric solution corresponding to flexural vibration of the plate.

Satisfying the homogeneous boundary conditions, we obtain the dispersion equation

$$f(\alpha, \Omega) \equiv \begin{vmatrix} a_1 \operatorname{sh}\beta_1 & a_2 \operatorname{sh}\beta_2 & a_3 \operatorname{sh}\beta_3\\ b_1 \operatorname{ch}\beta_1 & b_2 \operatorname{ch}\beta_2 & b_3 \operatorname{ch}\beta_3\\ \operatorname{ch}\beta_1 & \operatorname{ch}\beta_2 & \operatorname{ch}\beta_3 \end{vmatrix} = 0, \tag{2.4}$$

where

$$a_i = (-c_{13}\alpha^2 h_i + c_{33}e_i + e_{33})\beta_i;$$
$$b_i = \alpha\left[c_{44}(h_i\beta_i^2 + e_i) + e_{15}\right]. \tag{2.5}$$

The roots of this equation in fact determine the system of elementary solutions of the first kind. The actual solution of the problem (1.9) corresponding to a spectrum point α_k has the form

$$v_1^{(k)} = \sum_{i=1}^{3} A_i^{(k)} h_i^{(k)} \beta_i^{(k)} \operatorname{sh}(\beta_i^{(k)}\zeta),$$
$$v_2^{(k)} = \sum_{i=1}^{3} A_i^{(k)} e_i^{(k)} \operatorname{ch}(\beta_i^{(k)}\zeta), \tag{2.6}$$
$$v_3^{(k)} = \sum_{i=1}^{3} A_i^{(k)} \operatorname{ch}(\beta_i^{(k)}\zeta).$$

The dispersion equation (2.4) can be rewritten in the more usual form

$$\sum_{n=1}^{3} M_n \operatorname{th} \beta_n = 0. \tag{2.7}$$

§3. We first divide the roots of Eq. (2.4) into two groups: real and complex. The homogeneous solutions corresponding to the complex roots for small ε have the character of a boundary layer, while the homogeneous solutions corresponding to the real roots describe the interior mechanical and electrical fields. We investigate the behavior of the roots of Eq. (2.4) for small values of the parameter Ω, having used for this the perturbation method [7]. We write Eq. (1.9) in the form

$$L(-\alpha^2)v \equiv [L_0(-\alpha^2) + \Omega^2 L_1]v = 0. \tag{3.1}$$

The form of the operators L_0 and L_1 is established from comparison of the expressions (3.1), (1.9), and (1.6). From (3.1) it follows that the operator $L(-\alpha^2)$ can be considered as a perturbation of the infinite operator $L_0(-\alpha^2)$ by the finite operator $\Omega^2 L_1$. The spectrum of $L_0(-\alpha^2)$ is given by Eq. (2.4), if we put $\Omega = 0$ in it. For $\Omega = 0$ Eq. (2.4) has only one quadruple real root $\alpha = 0$ and a countable set of complex roots. For a number of materials of the systems RZT and TsTS, an asymptotic analysis shows that for $\Omega = 0$ in the upper half-plane there exist three branches of complex roots.

The first branch coincides with the imaginary axis; at the same time the asymptotic values of the roots are as follows:

$$\varkappa_k^0 = \frac{\frac{\pi}{2} + k\pi}{\theta_1}, \tag{3.2}$$

where $\theta_1^2 = (\beta_1/\alpha)^2 < 0$ is the real root of Eq. (2.3) for $\Omega = 0$. For materials of the system TsTS the points of intersection of the imaginary branches with the plane $\Omega = 0$ can also be determined from the expression

$$\varkappa_k^0 = \frac{k}{2} \frac{(15\varepsilon_{33})^{1/2}}{\sqrt{\varepsilon_{11} + \frac{e_{15}^2}{c_{44}}}},$$

the error here not exceeding 2%.

Two other branches are symmetrical relative to the imaginary axis. The position of the right branch and the location of the roots $\alpha_n^0 = z_n^0 + jy_n^0$ on it are described by the asymptotic relations

$$b_2 z_n^0 + a_2 y_n^0 = p_n; \quad a_2 z_n^0 - b_2 y_n^0 = r_n, \tag{3.3}$$

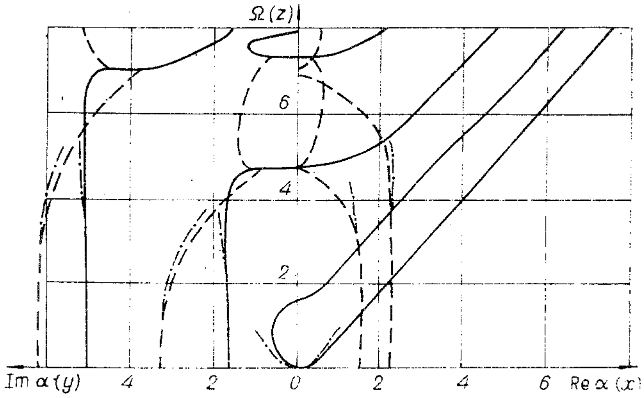

Fig. 1

where

$$p_n = (2n + 1/2)\pi/2;\; r_n = (1/2)\,\mathrm{arcsh}\,(\mathrm{Im}\,M_3/2\mathrm{Re}\,M_3 - \mathrm{Re}\,M_3/2\,\mathrm{Im}\,M_3).$$

Here $\theta_2 = \beta_2/\alpha = a_2 + jb_2$ is the root of the bicubic equation lying in the first quadrant.

The expressions (3.2), (3.3) already for k = n = 1 give the values of the roots not exceeding 1%. We can determine the character of behavior of the complex and imaginary roots for small Ω. Assuming that the quantity $\sigma = \Omega/|\alpha|$ is sufficiently small, we expand by powers of this quantity the roots of the bicubic equation (2.3):

$$\left(\frac{\beta_i}{\alpha}\right)^2 = \theta_i^2 - F_i\sigma^2, \tag{3.4}$$

where

$$F_i = f_i + jg_i = \frac{N_2\theta_i^4 - D_2\theta_i^2 + C_2}{3\theta_i^4 - 2N_2\theta_i^2 + D_3}.$$

Substituting β_1 from (3.4) into (3.2), after certain transformations we obtain the asymptotic approximations of the imaginary roots for low frequencies:

$$\varkappa_k(\Omega) = \varkappa_k^0 + \frac{f_1\Omega^2}{2\theta_1\left(\frac{\pi}{2} + k\pi\right)}. \tag{3.5}$$

Analogously substituting β_2 into (3.3), we obtain a system of two equations in z_n, y_n, which can be written in the form

$$\begin{aligned} &(a_2^2 - b_2^2)(z_n^2 - y_n^2) - 4a_2b_2z_ny_n - f_2\Omega^2 = r_n^2 - p_n^2,\\ &a_2b_2(z_n^2 - y_n^2) + (a_2^2 - b_2^2)z_ny_n - \frac{1}{2}g_2\Omega^2 = r_np_n. \end{aligned} \tag{3.6}$$

Substituting the actual values of the moduli of the ceramics, from the expressions (3.4)-(3.6) we can determine the character of dependence of the imaginary and complex roots of the dispersion equation for low frequencies. For the ceramics TsTS-19 they have the following form:

$$z_1 = 1.48(1-0.007\Omega^2);\quad y_1 = 3.47(1-0.02\Omega^2);\quad \varkappa_1 = 1.68(1+0.013\Omega^2);$$
$$z_2 = 2.11(1-0.004\Omega^2);\quad y_2 = 6.37(1-0.07\Omega^2);\quad \varkappa_2 = 5.04(1+0.002\Omega^2).$$

From the expressions we see that the quantities $\varkappa_k(\Omega)$, $z_k(\Omega)$, $y_k(\Omega)$ only slightly depend on the frequency. Therefore, for small frequencies they can be approximated by the roots from Eqs. (3.2), (3.3).

We consider the construction of asymptotic approximations of the roots in the neighborhood of the point $\alpha = 0$, $\Omega = 0$.

Since for the degenerate operator $\alpha = 0$ is a quadruple point of the spectrum, for the investigation of the spectrum of the operator $L(-\alpha^2)$ we use the branching methods [7]. The solution of the spectral problem (3.1) in this case is sought in the form

$$\alpha^2 = t_1^2\Omega + t_2^2\Omega^2 + \ldots, \quad v = v^{(0)} + \Omega v^{(1)} + \Omega^2 v^{(2)} + \ldots . \tag{3.7}$$

The substitution of (3.7) into (3.1) leads to a certain recursive system; integrating this system, we obtain

$$v_1^{(0)} = -w_0\zeta; \quad v_2^{(0)} = w_0; \quad v_3^{(0)} = 0; \tag{3.8}$$

$$v_2^{(1)} = t_1 w_0 \Delta_2 \frac{\zeta^2 - 1}{2} + w_1^0; \quad v_3^{(1)} = t_1 w_0 \Delta_1 \frac{\zeta^2 - 1}{2};$$

$$v_1^{(1)} = -t_1\left\{\left[\frac{c_{44} + c_{13}}{c_{44}}\Delta_2 + \frac{e_{31} + e_{15}}{c_{44}}\Delta_1 + \frac{c_{11}}{c_{44}}\right] w_0\left(\frac{\zeta^3}{6} - \frac{\zeta}{2}\right) + w_1^0\zeta; \quad w_1^0 = \text{const};\right.$$

$$t_1^4 = B^4; \quad B^4 = \frac{3c_{44}}{e_{31}\Delta_1 + c_{13}\Delta_2 + c_{11}}. \tag{3.9}$$

Here w_0 is an arbitrary constant which can be taken, for example, equal to unity.

Equation (3.9) has four roots:

$$t_{11} = B, \; t_{12} = -B, \; t_{13} = jB; \; t_{14} = -jB. \tag{3.10}$$

From the expression (3.10) we see that in the neighborhood of the point (0.0) there exist two real roots and two imaginary roots. The expression

$$\alpha = t_{11}\Omega^{1/2}(1 + 0(\Omega)) \tag{3.11}$$

describes the beginning of the first dispersion curve depicted in Fig. 1. The expression (3.11) guarantees 1% accuracy for $\Omega \leqslant \Omega_0 = 0.25$. For $\Omega > \Omega_0$ the first curve degenerates into a straight line with an angle of inclination (for TsTS-19) $V_R = \Omega/\alpha = 1.06$. The value of V_R corresponds to the phase velocity of Rayleigh waves for the given material.

We now consider the problem of constructing the dispersion curves for an arbitrary value of Ω.

We construct their asymptotics in the neighborhood of $\alpha = 0$. In Eqs. (2.3) and (2.4) we put $\alpha = 0$ and determine the values of Ω which are the beginning of the dispersion curves. We obtain two sets of values:

$$\Omega_m = m\pi/2, \; m = 1, 3, 5, \ldots,$$
$$\Omega_n = n\pi/k_0, \quad k_0 = \sqrt{\frac{c_{44}\varepsilon_{11}}{c_{33}\varepsilon_{33} + e_{33}^2}}, \quad n = 0, 1, 2, \ldots . \tag{3.12}$$

The values Ω_n and Ω_m found are resonances for the infinite layer.

To obtain the approximate roots for $|\alpha| \to 0$ or $\Omega \to m\pi/2$, we put

$$\Omega_m^2 \approx \frac{m^2\pi^2}{4} - R_m\alpha_m^2. \tag{3.13}$$

Taking into account the fact that α is small, from the combined analysis of Eqs. (2.3), (2.4) we obtain

$$R_m = -(8\beta_3/m^2\pi^2 \operatorname{tg}\beta_3)D + A_0, \tag{3.14}$$

where

$$A_0 = (N_2 - D_2 + C_1)/(2N_1 - D_1 - 3); \; \beta_3 = (m\pi/2)k_0;$$

$$D = \frac{\left[-c_{13}\frac{(c_{44} + c_{13})\,e_2 + (e_{31} + e_{15})}{c_{11} + c_{44}A_0} + c_{33}e_2 + e_{33}\right]\left[\frac{(c_{44} + c_{13})\,e_3 + (e_{31} + e_{15})}{c_{44}\left(1 - k_0^2\right)}k_0^2 + e_3\right]}{\left(c_{33}e_3 + e_{33}^2\right)\frac{(c_{44} + c_{13})\,e_2 + (e_{31} + e_{15})}{c_{11} + c_{44}A_0}k_0^2}.$$

Thus, for $R_m > 0$ imaginary roots occur for $\Omega_m > m\pi/2$ and real roots occur for $\Omega_m < m\pi/2$, while for $R_m < 0$ the converse takes place. Giving actual values to the moduli of the ceramics, we can calculate from (3.14) the quantities R_m. Thus, for the ceramics TsTS-19 $R_1 = 3.606$, $R_3 = -0.783$. Analogously, we can construct the asymptotics of the roots for $\Omega_n \to n\pi/k_0$. Going over to the isotropic case, we obtain from the expressions (3.13), (3.14) the well-known results [8]. Although these expressions have a small area of applicability, which decreases with the growth of m, they are of interest, since they are connected with the determination of portions of negative group velocities.

In Fig. 1 we have depicted the dispersion curves for the ceramics TsTS-19 obtained by numerically solving Eqs. (2.3), (2.4). The real and purely imaginary values are marked by the solid lines, with the purely imaginary roots located on the left of the origin of the coordinates.

The real and imaginary parts of the complex roots are marked by the dashed lines; here the Ox axis is the axis of real values of α, while the Oy axis is the axis of imaginary values of α. Along the Oz axis we have set off the values of the dimensionless frequency Ω. Dashed-dot lines are used to denote curves corresponding to the asymptotic approximations of the roots. It is seen that for low frequencies the imaginary and complex roots of the dispersion equation (2.4) vary little in comparison with their asymptotic values $\varkappa_k(\Omega)$, $z_k(\Omega)$, $y_k(\Omega)$ from the expressions (3.5), (3.6); here the frequency range of applicability of these expressions increases with the growth of the ordinal number k. For k = 1 the error does not exceed 5% for $\Omega \leqslant 2$, while for k = 2 it is not exceeded for $\Omega \leqslant 4$.

Consideration of the piezoeffect does not lead to any qualitative changes for the real and complex curves in comparison with the isotropic problem. They are only displaced by a small amount which depends on the choice of the constants of the piezoceramics. An analogous result for the real roots has been obtained in [6]. However, the emergence of an almost vertical branch corresponding to the purely imaginary roots can be explained only by the piezoelectric coupling. On the basis of the dispersion curves, for each Ω we can determine the phase and group velocities of propagation of a disturbance in a plane waveguide.

The latter, as we know, characterizes energy transfer. Here it is of interest to investigate the distribution of the energy being transferred in the case of a fixed frequency over individual modes corresponding to the dispersion curves. However, such a problem can be dealt with, having solved the particular boundary-value problem, i.e., having determined the functions $m_k(\xi, \eta)$, $n_k(\xi, \eta)$ (the moduli of the piezoceramics TsTS-19 are taken from [9]).

LITERATURE CITED

1. H. F. Tiersten and R. D. Mindlin, "Forced vibrations of piezoelectric crystal plates," Quart. Appl. Math., 20, 107 (1962).
2. R. D. Mindlin, "High-frequency vibrations of piezoelectric crystal plates," Intern. J. Solids Struct., 8, No. 7 (1972).
3. V. Lypacewicz and L. Filipipezyncki, "Vibrations of piezoelectric ceramic transducers loaded mechanically," Proc. Vibr. Probl., 11, No. 3 (1972).
4. O. K. Aksentyan and I. I. Vorovich, "Stress state of a plate of small thickness," Prikl. Mat. Mekh., 28, No. 6 (1963).
5. N. Ya. Kotsarenko et al., "Electronic damping and reinforcement of Lamb waves in piezosemiconductors," Ukrainsk. Fiz. Zh., 16, No. 10 (1971).
6. N. F. Ivina and B. A. Kasatkin, "Normal waves in an anisotropic piezoactive waveguide," Defektoskopiya, No. 4 (1975).
7. M. M. Vainberg and V. A. Trenogin, The Branching Theory of Solutions of Nonlinear Equations [in Russian], Nauka, Moscow (1969).
8. U. K. Nigul, "On the roots of the Lamb equation for deformation of a plate which is antisymmetric about the middle surface," Izv. Akad. Nauk ÉstSSR, 12, No. 3 (1963).
9. E. G. Smazhevskaya and N. B. Fel'dman, Piezoelectric Ceramics [in Russian], Sov. Radio, Moscow (1971).